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On the slow motion of a self-propelled rigid body in a viscous incompressible fluid

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Abstract

In this paper we study the Stokes approximation of the self-propelled motion of a rigid body in a viscous liquid that fills all the three-dimensional space exterior to the body. We prove the existence and uniqueness of strong solution to the coupled systems of equations describing the motion of the system body–liquid, for any time and any regular distribution of velocity on the boundary of the body. For the corresponding stationary problem we derive L^p-estimates for the solution in terms of the data. Finally, we prove that every steady solution is attainable as the limit, when $t \to \infty$, of an unsteady self-propelled solution which starts from rest.

1. Introduction

Important practical aims, like improved locomotion of vehicles through water, air and space, and a better understanding of the functioning and evolution of locomotion of animals within their surrounding medium, have underlain the development and study of mathematical models describing the motion by self-propulsion of a body in an infinitely extended fluid [18,19].

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In a pure motion by self-propulsion the total net force and torque, external to the system body–fluid, acting on the body are zero. The forward force (thrust) that makes the body move is generated by the body itself and the motion is due to the interaction of the body's external surface and the fluid in which it is immersed. The hydrodynamical mechanism of self-propulsion is different for macroscopic and microscopic bodies [18,29]. Large objects which propel themselves make use of inertia in the surrounding fluid. Their thrust can be produced by muscular action and change of shape, as in animal locomotion [12], or can be provided by mechanical propulsion systems, as in an airplane, rocket or submarine [19]. However, this conception of self-propulsion cannot be transferred to microscopic organisms since the inertia forces are negligible compared with the forces due to viscosity [29]. For instance, many ciliated micro-organisms, like *Opalina* and *Paramecium*, propel themselves by moving small hair-like organelles, called cilia, which cover its external surface [27]. A different mechanism of propulsion is used, for example, by spermatozoa, which move a thin tail down which the organism send waves of lateral displacement [29]. The influence of inertia in the motion of a self-propelled body is experimentally shown by Taylor [30], comparing the velocity of a mechanical fish, which moves its tail symmetrically, in water and in a very viscous liquid: the fish moves in water, but makes no progress in a very viscous liquid.

Motivated by important problems in medical and environmental science [18], the theoretical study of micro-organism motion, was initiated by Taylor [29], and extended by Lighthill [18]. Several models and frameworks [2,14,25], see also the survey paper [16], have been proposed to study self-propulsion of micro-organisms, which is the typical example of self-propulsion at zero Reynolds number.

In this paper, we shall be interested in the self-propulsion of a rigid body at vanishing Reynolds number. Since the shape of the body is constant during the motion, the thrust is produced either because the body generates a nonzero momentum flux through its boundary, or/and because it moves portions of its boundary [10]. As it was already mentioned, in the limit of zero Reynolds number, the importance of inertia in determining the motion of the fluid, and consequently, the motion of the body, becomes negligible. The motion of the body is therefore completely determined by its geometry and by the distribution of velocity on its boundary. In fact, it has been shown in [10] that, in the steady case, the motion of the body can be completely decoupled from that of the liquid, and the method used in [9] can also be extended to unsteady self-propelled motion to separate the motions of the body and the liquid.

To better explain our results, let us give a mathematical formulation of the problem. We consider a rigid body, represented by a compact set $\mathcal{B} \subset \mathbb{R}^3$, moving in a viscous liquid $\mathcal{L}$ which occupies the region $\mathcal{D} = \mathbb{R}^3 \setminus \mathcal{B}$ exterior to the body.

In the situation of vanishing Reynolds number (see, e.g., [10,17]), the motion of $\{\mathcal{B}, \mathcal{L}\}$ is described by the coupled systems of equations

$$\begin{cases} \left.\begin{array}{l} \partial_t(v+V) = \operatorname{div} T(v,p) \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } \mathcal{D} \times]0,T[\,, \\ v = v_* \quad \text{at } \Sigma \times]0,T[\,, \\ \lim_{|x|\to\infty}(v(x,t)+V(x,t)) = 0 \quad \text{for } t \in]0,T[\,, \\ \left.\begin{array}{l} m\frac{d\xi}{dt} = -\int_\Sigma T(v,p)\cdot n \\ I\cdot\frac{d\omega}{dt} = -\int_\Sigma x \times T(v,p)\cdot n \end{array}\right\} \quad \text{in }]0,T[\,, \\ v(x,0) = v_0(x), \quad x \in \mathcal{D}, \\ \xi(0) = \xi_0, \qquad \omega(0) = \omega_0. \end{cases} \tag{1.1}$$

The quantities $v = v(x,t)$ and $p = p(x,t)$ represent the velocity and pressure associated to each particle of $\mathcal{L}$, in a frame attached to $\mathcal{B}$ with the origin of coordinates coinciding with the center of mass of $\mathcal{B}$, and $T(v,p)$ is the stress tensor, defined by

$$T_{ij}(v,p) = 2D_{ij}(v) - p\delta_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - p\delta_{ij}.$$

The field $V(x,t) = \xi(t) + \omega(t) \times x$ represents the velocity of $\mathcal{B}$, which is an unknown in our problem. In Eqs. $(1.1)_{5,6}$ the positive constant m is the mass of $\mathcal{B}$ and I is its inertia tensor. Recall that

$$I_{ij} = \int_{\mathcal{B}} \rho(x)\big(|x|^2\delta_{ij} - x_i x_j\big)\,dx,$$

and I is symmetric and positive definite (see, e.g., [4]). The distribution of velocity v_* on Σ represents the thrust, responsible for the motion of the body.

If $\{\mathcal{B}, \mathcal{L}\}$ performs a steady motion then system (1.1) takes the form

$$\begin{cases} \left.\begin{array}{l} \operatorname{div} T(v,p) = 0 \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } \mathcal{D}, \\ v = v_* \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}(v(x)+V(x)) = 0, \\ \int_\Sigma T(v,p)\cdot n = 0, \\ \int_\Sigma x \times T(v,p)\cdot n = 0. \end{cases} \tag{1.2}$$

Such approximations can be adopted either when the viscosity of the fluid is large or the maximum of the velocity distribution at Σ is small or when the size of $\mathcal{B}$ is small (see [10]).

Before explaining the objectives and results of this paper, let us briefly refer to the literature on the mathematical analysis of motion by self-propulsion of a rigid body in an infinite Navier–Stokes fluid. In [6,20,21] the asymptotic properties of steady flow past a self-propelled body moving with purely translational velocity are investigated. The existence of such solutions was first established for very particular shapes, like balls and cylinders in [22–24] and for a symmetric body around an axis in [8]. Considering the general form of a rigid body motion, with the rotation of the body taken into account, in [10], the existence of steady self-propelled solutions was proved for a body with arbitrary geometry, with a detailed study of the cases of zero and nonzero Reynolds number. In [26], we prove the existence of a weak solution to the general unsteady nonlinear problem and the attainability of steady purely translational self-propelled motion for a symmetric body.

In this paper, our objectives will be to solve (1.1), for any $0 < T \leqslant \infty$ and any regular and compatible boundary and initial data, and to prove that any (sufficiently regular) steady solution (1.2) is attainable from rest, as the asymptotic limit $t \to \infty$ of a nonstationary self-propelled motion. Moreover, we derive the summability properties and corresponding estimates in terms of the data for the steady solutions (1.2).

The existence and uniqueness of global strong solution to (1.1) will be established using the theory of analytic semigroups of linear operators in Hilbert spaces. We introduce an appropriate functional setting for the problem, which allows us to study the problem in the L^2-framework of energy estimates, and we study an (unbounded) linear operator associated with the equations in these functions spaces. The key point of our approach is the decomposition of the space $\mathcal{L}(\mathcal{D})$ (Lemma 3.2) introduced in [26], and the operator A defined in Section 4. We then prove (Theorem 4.2) the existence and uniqueness of a strong solution to the following problem:

$$\begin{cases} \left.\begin{array}{l} \partial_t(u+V) = \operatorname{div} T(u,p) + f + \overline{f} \\ \operatorname{div} u = 0 \end{array}\right\} \quad \text{in } \mathcal{D}\times]0,T[\,, \\ u = 0 \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}(u(x,t)+V(x,t)) = 0, \qquad t \in]0,T[\,, \\ \left.\begin{array}{l} m\frac{d\xi}{dt} = -\int_\Sigma T(u,p)\cdot n + mf_1 \\ I\cdot\frac{d\omega}{dt} = -\int_\Sigma x\times T(u,p)\cdot n + I\cdot f_2 \end{array}\right\} \quad \text{in }]0,T[\,, \\ u(x,0) = u_0(x), \qquad x\in\mathcal{D}, \\ \xi(0) = \xi_0, \qquad \omega(0) = \omega_0, \end{cases} \tag{1.3}$$

where $\overline{f}(x) = f_1 + f_2 \times x$. Existence and uniqueness results to problem (1.1) (see Theorem 5.1) will be obtained after reducing (1.1) to a problem of the form (1.3), by extending v_* to $\mathcal{D}$.

The attainability of steady solution (Theorem 5.3) will be established by considering a perturbation problem for the steady state (1.2), which has the form (1.3). Using well known estimates for the solution to an abstract Cauchy problem and for the Stokes problem in exterior domains, we then show that the perturbation tends to zero, and derive the corresponding order of decay

$$\left\|\partial_t(v+V)(t)\right\|_2, \left|\frac{d\xi}{dt}\right|, \left|\frac{d\omega}{dt}\right| = O(t^{-1}),$$

$$\begin{aligned}
&\|D(v(t)-v_\infty)\|_2, \ |\xi(t)-\xi_\infty|, \ |\omega(t)-\omega_\infty| = O(t^{-1/2}),\\
&|v(t)-v_\infty|_{2,2}, \ |p(t)-p_\infty|_{1,2} = O(t^{-1/2}),\\
&\sup_{x\in\mathcal{D}}|v(x,t)-V_\infty(x,t)| = O(t^{-1/2}),
\end{aligned}$$

as $t\to\infty$. To get these results, we need an estimate for the L^2-norm of the solution to the steady problem (1.2). Like the steady flow of a Navier–Stokes liquid past a self-propelled body moving with constant velocity [6,20,21], the solution to (1.2) possesses far field decay properties different from the corresponding towed solution with the same rigid body velocity (see Theorem 5.2).

In [9], Galdi solves problem (1.3) by a different method, decoupling the equations of motion of the body from those of the fluid. However, the solution is found in a different space. For a similar approach, in a complete different context, see Grobbelaar-Van Dalsen and Sauer [13].

The plan of the paper is the following. In Section 2 we introduce some notations and auxiliary results, namely, on the solenoidal extension of a function defined on the boundary of the body and on the classical steady Stokes problem in exterior domain. In Section 3 we introduce the functional setting for our problem and in Section 4 we prove existence and uniqueness of solution to problem (1.3). Section 5 is dedicated to the self-propelled motion in the Stokes approximation. In Section 5.1, we construct an extension of v_* to the liquid domain and solve problem (1.1); after, in Section 5.2, we study the stationary problem (1.2), in particular, the summability properties of solutions and corresponding estimates in terms of the data. Finally, in Section 5.3, we prove the attainability of self-propelled steady motions.

2. Notation and auxiliary results

Throughout the paper we shall use the same font style to denote scalar, vector and tensor-valued functions. We will follow Einstein's summation convention.

For any open set $\mathcal{A}\subset\mathbb{R}^3$, $L^q(\mathcal{A})$, $1\leqslant q\leqslant\infty$, and $W^{m,q}(\mathcal{A})$, $m\geqslant 0$, denote the usual Lebesgue and Sobolev spaces, respectively, with norms $\|\cdot\|_{q,\mathcal{A}}$ and $\|\cdot\|_{m,q,\mathcal{A}}$. We shall write $u\in W^{m,q}_{\mathrm{loc}}(\overline{\mathcal{A}})$ to mean $u\in W^{m,q}(\mathcal{A}')$, for any bounded domain $\mathcal{A}'\subset\mathcal{A}$. By $W^{m-1/q,q}(\partial\mathcal{A})$ we denote the trace space on $\partial\mathcal{A}$ for functions from $W^{m,q}(\mathcal{A})$, equipped with the norm $\|\cdot\|_{m-1/q,q,\partial\mathcal{A}}$. Since we will deal with a problem in an exterior domain, it is natural to consider the homogeneous Sobolev spaces $D^{m,q}(\mathcal{A})$, $m\geqslant 0$, $1\leqslant q\leqslant\infty$, defined by

$$D^{m,q}(\mathcal{A}) := \{u\in L^1_{\mathrm{loc}}(\mathcal{A}):\ D^l u\in L^q(\mathcal{A}),\ |l|=m\}$$

with associated seminorm

$$|u|_{m,q,\mathcal{A}} := \left(\sum_{|l|=m} \int_{\mathcal{A}} |D^l u|^q \right)^{1/q}.$$

Whenever confusion will not arise, we shall omit the subscript $\mathcal{A}$ in the previous norms and seminorms. For details on these functions spaces see the books [1] and [7].

Let X be a Banach space with norm $\|\cdot\|_X$. For $p \in [1,\infty[$ and $T > 0$, we denote by $L^p(0,T;X)$ the space of measurable functions $u:]0,T[\to X$ such that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

If I is a real interval, we denote by $C^m(I;X)$, $m \geqslant 0$, the space of all continuous functions on the interval I with values in X which have continuous derivatives up to the order m included. By $H_{\mathrm{loc}}([0,T[\,;X)$ we denote the space of functions $f: [0,T[\to X$ which are locally Hölder continuous; that is, for each $\tau \in]0,T[$, there are numbers $K = K(\tau) > 0$, $\alpha = \alpha(\tau) \in]0,1]$ such that

$$\big\| f(t) - f(s) \big\|_X \leqslant K|t-s|^\alpha \quad \text{for } 0 \leqslant s,\ t \leqslant \tau,$$

and $H^1_{\mathrm{loc}}([0,T[\,;X)$ will be the subspace of $H_{\mathrm{loc}}([0,T[\,;X)$ of functions whose first derivative is in $H_{\mathrm{loc}}([0,T[\,;X)$.

In all that follows, $\mathcal{B} \subset \mathbb{R}^3$ is a connected compact set, representing the rigid body, and $\mathcal{D} = \mathbb{R}^3 \setminus \mathcal{B}$ is the domain occupied by the liquid $\mathcal{L}$. By Σ we denote the boundary of $\mathcal{B}$ and $\mathcal{D}$, and assume Σ to be of class C^2. The unit normal to Σ pointing into $\mathcal{B}$ is denoted by n. We locate the origin of coordinates in $\mathcal{B}$ and denote by $\delta(\mathcal{B})$ the diameter of $\mathcal{B}$. For each $R > \delta(\mathcal{B})$, we set

$$\mathcal{D}_R := \mathcal{D} \cap \big\{ x \in \mathbb{R}^3\colon |x| < R \big\}, \qquad \mathcal{D}^R := \mathcal{D} \setminus \big\{ x \in \mathbb{R}^3\colon |x| \leqslant R \big\}.$$

We shall need to extend functions on Σ to $\mathcal{D}$. The next lemma concerns extensions with compact support.

Lemma 2.1. *Let $v_* \in W^{3/2,2}(\Sigma)$ satisfy the condition*

$$\int_\Sigma v_* \cdot n = 0.$$

Then there exists $\tilde{v}_* \in W^{2,2}(\mathcal{D})$ with compact support such that

$$\begin{cases} \operatorname{div} \tilde{v}_* = 0 & \text{in } \mathcal{D}, \\ \tilde{v}_* = v_* & \text{at } \Sigma, \end{cases}$$

and

$$\|\tilde{v}_*\|_{2,2} \leqslant C(\mathcal{B})\|v_*\|_{3/2,2,\Sigma}.$$

Proof. Fix $R > \delta(\mathcal{B})$. There exists $\overline{v_*} \in W^{2,2}(\mathcal{D}_{3R})$ that verifies

$$\begin{cases} \operatorname{div} \overline{v_*} = 0 & \text{in } \mathcal{D}_{3R}, \\ \overline{v_*} = v_* & \text{at } \Sigma, \\ \overline{v_*} = 0 & \text{at } \partial B_{3R}, \end{cases}$$

and the estimate

$$\|\overline{v_*}\|_{2,2,\mathcal{D}_{3R}} \leqslant C(\mathcal{B}, R)\|v_*\|_{3/2,2,\Sigma}.$$

Now, let $\psi \in C^\infty(\mathcal{D})$ be such that $\psi = 1$ in $\mathcal{D}_R$ and $\psi = 0$ in $\mathcal{D}^{2R}$. Since

$$\operatorname{div}(\psi \overline{v_*}) = \overline{v_*} \cdot \nabla\psi + \psi \operatorname{div} \overline{v_*} = \overline{v_*} \cdot \nabla\psi$$

we have $\operatorname{div}(\psi \overline{v_*}) \in W_0^{1,2}(\mathcal{D}_{3R})$ and, by Theorem III.3.2 in [7], there exists

$$\begin{cases} \operatorname{div} w = \operatorname{div}(\psi \overline{v_*}) & \text{in } \mathcal{D}_{3R}, \\ w \in W_0^{2,2}(\mathcal{D}_{3R}). \end{cases}$$

Moreover, w verifies the estimate

$$\|w\|_{2,2,\mathcal{D}_{3R}} \leqslant C(\mathcal{B})\|v_*\|_{3/2,2,\Sigma}.$$

We define $\tilde{v}_*$ by

$$\tilde{v}_* = \psi \overline{v_*} - w. \qquad \square$$

Remark. We emphasize that the extension $\tilde{v}_*$ obtained in the previous theorem can be constructed using explicit representation formulas, see Chapter III of [7]. If $v_* = v_*(x,t)$ with $\int_\Sigma v_*(x,t) \cdot n_x \, d\sigma_x = 0$ and $\partial_t v_* \in W^{1/2,2}(\Sigma)$, for almost all $t \in]0,T[$, then it is easily seen that $\partial_t \tilde{v}_* \in W^{1,2}(\mathcal{D})$, $\operatorname{div}(\partial_t \tilde{v}_*) = 0$ and

$$\|\partial_t \tilde{v}_*\|_{1,2,\mathcal{D}} \leqslant C(\mathcal{B})\|\partial_t v_*\|_{1/2,2,\Sigma},$$

for almost all $t \in]0,T[$.

For each $i \in \{1,\ldots,6\}$, set

$$\tilde{e}_i = \begin{cases} e_i, & i = 1,2,3, \\ e_{i-3} \times x, & i = 4,5,6, \end{cases}$$

where e_i is the ith vector of the canonical basis of $\mathbb{R}^3$ and consider the fields (H_i, P_i) $(i = 1,\ldots,6)$ which are solutions to the Stokes problems

$$\begin{cases} \left.\begin{array}{l} \operatorname{div} T(H_i, P_i) = 0 \\ \operatorname{div} H_i = 0 \end{array}\right\} \quad \text{in } \mathcal{D}, \\ H_i = 0 \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}(H_i + \tilde{e}_i)(x) = 0. \end{cases} \tag{2.1}$$

Lemma 2.2. *For each $i \in \{1,\dots,6\}$, problem* (2.1) *has a unique solution (H_i, P_i) such that $H_i, P_i \in C^\infty(\mathcal{D})$,*

$$(H_i + \tilde{e}_i, P_i) \in \big(L^s(\mathcal{D}) \cap D^{1,r}(\mathcal{D}) \cap D^{2,\tau}(\mathcal{D})\big) \times \big(L^r(\mathcal{D}) \cap D^{1,\tau}(\mathcal{D})\big),$$

for $s \in]3,\infty]$, $r \in]3/2,\infty]$, $\tau \in]1,\infty[$, and (H_i, P_i) obeys the following estimates:

$$\begin{aligned}&\big\|\big(1+|x|\big)(H_i+\tilde{e}_i)\big\|_\infty + \big\|\nabla(H_i+\tilde{e}_i)\big\|_\infty + \|P_i\|_\infty \\ &\quad + \|H_i+\tilde{e}_i\|_s + |H_i+\tilde{e}_i|_{1,r} + |H_i|_{2,\tau} + \|P_i\|_r + |P_i|_{1,\tau} \leqslant C,\end{aligned}$$

with $C = C(\mathcal{B}, s, r, \tau)$.

Proof. See [7, Chapter V]. □

Now, for each $i \in \{1,\dots,6\}$ let (u_i, p_i) be the solution of the following Stokes resolvent problem:

$$\begin{cases} \left.\begin{array}{l}\operatorname{div} T(u_i, p_i) = u_i + \tilde{e}_i \\ \operatorname{div} u_i = 0\end{array}\right\} \quad \text{in } \mathcal{D}, \\ u_i = 0 \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}(u_i + \tilde{e}_i)(x) = 0. \end{cases} \tag{2.2}$$

Lemma 2.3. *For each $i \in \{1,\dots,6\}$, problem* (2.2) *has a unique solution $(u_i, p_i) \in D^{2,2}(\mathcal{D}) \times D^{1,2}(\mathcal{D})$ such that $u_i + \tilde{e}_i \in W^{2,2}(\mathcal{D})$.*

Proof. Since

$$\int_\Sigma \tilde{e}_i \cdot n = 0,$$

by Lemma 2.1, there exists $\psi_i \in W^{2,2}(\mathcal{D})$ with compact support such that

$$\begin{cases} \operatorname{div} \psi_i = 0, & \text{in } \mathcal{D}, \\ \psi_i = \tilde{e}_i, & \text{at } \Sigma. \end{cases}$$

Then, by Theorem 2.1 in [5], there exists a unique pair $(U_i, p_i) \in W^{2,2}(\mathcal{D}) \times D^{1,2}(\mathcal{D})$ solving

$$\begin{cases} \left.\begin{array}{l}\operatorname{div} T(U_i, p_i) = U_i \\ \operatorname{div} U_i = 0\end{array}\right\} \quad \text{in } \mathcal{D}, \\ U_i = \tilde{e}_i \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty} U_i(x) = 0. \end{cases}$$

Therefore, $(u_i, p_i) = (U_i - \tilde{e}_i, p_i)$ is the unique solution to (2.2) that verifies $(u_i, p_i) \in D^{2,2}(\mathcal{D}) \times D^{1,2}(\mathcal{D})$ and $u_i + \tilde{e}_i \in W^{2,2}(\mathcal{D})$. □

Lemma 2.4. *If $v_* \in W^{2-1/q,q}(\Sigma)$, $3 < q < \infty$, the problem*

$$\begin{cases} \left.\begin{aligned} &\operatorname{div} T(u,\pi) = 0 \\ &\operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}, \\ u = v_* \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{2.3}$$

has a unique solution (u,π) such that $u,\pi \in C^\infty(\mathcal{D})$,

$$(u,\pi) \in \left(L^s(\mathcal{D}) \cap D^{1,r}(\mathcal{D}) \cap D^{2,\tau}(\mathcal{D})\right) \times \left(L^r(\mathcal{D}) \cap D^{1,\tau}(\mathcal{D})\right)$$

for $s \in]3,\infty]$, $r \in]3/2,\infty]$, $\tau \in]1,q]$, and (u,π) obeys the following estimates:

$$\begin{aligned} &\left\|\left(1+|x|\right)u\right\|_\infty + \|\nabla u\|_\infty + \|\pi\|_\infty \\ &\quad + \|u\|_s + |u|_{1,r} + |u|_{2,\tau} + \|\pi\|_r + |\pi|_{1,\tau} \leqslant C\|v_*\|_{2-1/q,q,\Sigma}, \end{aligned}$$

with $C = C(\mathcal{B}, q, s, r, \tau)$.

Proof. See [7, Chapter V]. □

Next we recall some results about theory of semigroups of linear operators.

Theorem 2.1. *Let $\mathbb{H}$ be a real Hilbert space and let A be a linear operator in $\mathbb{H}$. If A is symmetric and maximal dissipative then A is the infinitesimal generator of an analytic semigroup on $\mathbb{H}$.*

Proof. See, e.g., [28] and [3]. □

Consider the Cauchy problem

$$\begin{cases} \frac{du}{dt} = Au + f, \quad \text{in }]0,T[\,, \\ u(0) = u_0, \end{cases} \tag{2.4}$$

where $0 < T \leqslant \infty$. We have the following (see [11] and [28])

Theorem 2.2. *Let $\mathbb{H}$ be a real Hilbert space. Suppose that A generates an analytic semigroup $\{T(t)\colon\ t \geqslant 0\}$ on $\mathbb{H}$. Let $f \in H_{\mathrm{loc}}([0,T[\,;\mathbb{H})$ and $u_0 \in \mathbb{H}$. Then* (2.4) *has a unique solution $u \in C([0,T[\,;\mathbb{H}) \cap C^1(]0,T[\,;\mathbb{H})$ such that $u(t) \in D(A)$, for each $t \in]0,T[$, and $Au \in C(]0,T[\,;\mathbb{H})$.*

Suppose that $\{T(t)\colon\ t \geqslant 0\}$ is a semigroup of contractions on $\mathbb{H}$. Then, if $f \in L^1(0,T;\mathbb{H})$, it holds

$$\|u(t)\|_{\mathbb{H}} \leqslant \|u_0\|_{\mathbb{H}} + \|f\|_{L^1(0,T;\mathbb{H})}, \quad \textit{for all } t \in]0,T[\,,$$

and, if $f \equiv 0$, then

$$\left\|\frac{du}{dt}(t)\right\|_{\mathbb{H}} \leqslant \frac{1}{t}\|u_0\|_{\mathbb{H}}, \quad \textit{for all } t > 0.$$

3. Functional setting for the problem

We denote by $\mathcal{R}$ the set of all velocity fields in a rigid motion,

$$\mathcal{R}=\left\{u\in C^\infty(\mathbb{R}^3)\colon\ u(x)=u_1+u_2\times x\equiv\bar{u}(x),\ u_1,u_2\in\mathbb{R}^3\right\}.$$

Consider the linear space

$$\mathcal{V}(\mathcal{D})=\left\{u\in W^{1,2}_{\mathrm{loc}}(\overline{\mathcal{D}})\colon\ u=0 \text{ at } \Sigma,\ \operatorname{div}u=0 \text{ in } \mathcal{D} \text{ and } D(u)\in L^2(\mathcal{D})\right\}$$

which is an Hilbert with respect to the scalar product

$$(u,v)_{\mathcal{V}}=\int_{\mathcal{D}} D(u):D(v).$$

Recall the following result (see [10]).

Lemma 3.1. *Let $u\in\mathcal{V}(\mathcal{D})$. Then, there exists a uniquely determined $\bar{u}\in\mathcal{R}$ such that*

$$\lim_{|x|\to\infty}\int_{S^2}\left|(u+\bar{u})\big(|x|,\sigma\big)\right|^2 d\sigma=0,$$

where S^2 is the unit sphere centered in 0. Moreover, $\nabla(u+\bar{u})\in L^2(\mathcal{D})$ and there exists a positive constant $C=C(\mathcal{B})$ such that

$$|u_1|+|u_2|+\left\|\nabla(u+\bar{u})\right\|_2\leqslant C\|D(u)\|_2.$$

Let $\mathcal{L}(\mathcal{D})$ be the linear space defined by

$$\mathcal{L}(\mathcal{D})=\left\{u\in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})\colon\ \exists^1\bar{u}\in\mathcal{R},\ u+\bar{u}\in L^2(\mathcal{D})\right\}.$$

It is clear that $\mathcal{R}$, $L^2(\mathcal{D})\subset\mathcal{L}(\mathcal{D})$. The space $\mathcal{L}(\mathcal{D})$ is an Hilbert space with respect to the scalar product

$$(u,v)_{\mathcal{L}}=\int_{\mathcal{D}}(u+\bar{u})\cdot(v+\bar{v})+mu_1\cdot v_1+u_2\cdot I\cdot v_2$$

whose associated norm is

$$\|u\|_{\mathcal{L}}=\left(\|u+\bar{u}\|_2^2+m|u_1|^2+u_2\cdot I\cdot u_2\right)^{1/2}. \tag{3.1}$$

Consider the closed linear subspaces of $\mathcal{L}(\mathcal{D})$

$$\mathcal{H}(\mathcal{D})=\left\{u\in\mathcal{L}(\mathcal{D})\colon\ \operatorname{div}u=0 \text{ and } u\cdot n_{|\Sigma}=0\right\},$$

$$\mathcal{G}(\mathcal{D})=\Bigg\{u\in\mathcal{L}(\mathcal{D})\colon\ \exists p\in D^{1,2}(\mathcal{D}),\ u+\bar{u}=\nabla p,\ \text{with}\\ u_1=-\frac{1}{m}\int_{\Sigma}pn,\ u_2=-I^{-1}\cdot\Bigg(\int_{\Sigma}px\times n\Bigg)\Bigg\}.$$

Notice that in the space $\mathcal{G}(\mathcal{D})$ the function p is defined up to an additive constant C, because $\nabla p = \nabla(p + C)$ and

$$\int_{\Sigma} n = \int_{\Sigma} x \times n = 0,$$

which will allows us to rescale the pressure p in such a way that $\int_{\mathcal{D}_R} p = 0$, for fixed $R > \delta(\mathcal{B})$.

The following decomposition of $\mathcal{L}(\mathcal{D})$ is valid.

Lemma 3.2. $\mathcal{L}(\mathcal{D}) = \mathcal{H}(\mathcal{D}) \oplus \mathcal{G}(\mathcal{D})$.

Proof. See [26]. □

Lemma 3.3. *Let* $(u, p) \in \mathcal{H}(\mathcal{D}) \cap \mathcal{V}(\mathcal{D}) \cap D^{2,2}(\mathcal{D}) \times D^{1,2}(\mathcal{D})$ *and let* $\bar{u}(x) = u_1 + u_2 \times x$ *be such that* $u + \bar{u} \in W^{2,2}(\mathcal{D})$. *Then there exist positive constants* $C_i = C_i(\mathcal{B})$, $i = 1, 2$, *such that*

$$|u|_{2,2} + |p|_{1,2} \leqslant C_1\big(\big\|\operatorname{div} T(u, p)\big\|_2 + \|u + \bar{u}\|_2 + |u_1| + |u_2|\big),$$
$$|u|_{2,2} + |p|_{1,2} \leqslant C_2\big(\big\|\operatorname{div} T(u, p)\big\|_2 + \|D(u)\|_2\big).$$

Proof. For fixed $R > \delta(\mathcal{B})$, if we normalize p by the condition

$$\int_{\mathcal{D}_R} p = 0,$$

by Lemma IV.1.1 in [7], it holds

$$\begin{aligned}\|p\|_{2,\mathcal{D}_R} &\leqslant C(\mathcal{B}, R)\big(\big\|\operatorname{div} T(u, p)\big\|_{-1,2,\mathcal{D}_R} + |u|_{1,2,\mathcal{D}_R}\big)\\ &\leqslant C(\mathcal{B}, R)\big(\big\|\operatorname{div} T(u, p)\big\|_2 + |u|_{1,2,\mathcal{D}_R}\big).\end{aligned}$$

Then, by Lemma V.4.2 in [7], we have

$$|u|_{2,2} + |p|_{1,2} \leqslant C(\mathcal{D}, R)\big(\big\|\operatorname{div} T(u, p)\big\|_2 + |u|_{1,2,\mathcal{D}_R}\big).$$

Now, by Ehrling inequality (see, e.g., [7]),

$$\begin{aligned}|u|_{1,2,\mathcal{D}_R} &\leqslant C(\mathcal{D}_R)\left(\|u\|_{2,\mathcal{D}_R} + \frac{1}{2}|u|_{2,2}\right)\\ &\leqslant C(\mathcal{B}, R)\left(\|u + \bar{u}\|_2 + |u_1| + |u_2| + \frac{1}{2}|u|_{2,2}\right)\end{aligned}$$

from which we obtain the first inequality (choosing, e.g., $R = 2\delta(\mathcal{B})$). Taking into account Lemma 3.1, we have

$$|u|_{1,2,\mathcal{D}_R} \leqslant C(\mathcal{D}_R)\big(\big\|\nabla(u + \bar{u})\big\|_{2,\mathcal{D}_R} + |u_2|\big) \leqslant C(\mathcal{B}, R)\|D(u)\|_2$$

and we get the second inequality. □

4. Resolution of problem (1.3)

We define a 6×6 matrix M by

$$M_{ij} = \int_{\Sigma} \tilde{e}_j \cdot T(u_i, p_i) \cdot n, \quad i, j = 1, \ldots, 6,$$

where (u_i, p_i) solves (2.2).

Lemma 4.1. *M is symmetric and positive definite.*

Proof. Multiplying both sides of (2.2) by $u_j + \tilde{e}_j$ and integrating by parts, we get

$$M_{ij} = \int_{\mathcal{D}} (u_j + \tilde{e}_j) \cdot (u_i + \tilde{e}_i) + 2 \int_{\mathcal{D}} D(u_j) : D(u_i)$$

and this shows that M is symmetric.

To prove that M is positive definite, we first notice that

$$\begin{aligned} \alpha_i M_{ij} \alpha_j &= \int_{\mathcal{D}} (\alpha_j u_j + \alpha_j \tilde{e}_j) \cdot (\alpha_i u_i + \alpha_i \tilde{e}_i) + 2 \int_{\mathcal{D}} D(\alpha_j u_j) : D(\alpha_i u_i) \\ &= \|\alpha_i u_i + \alpha_i \tilde{e}_i\|_2^2 + 2\|D(\alpha_i u_i)\|_2^2 \geqslant 0 \end{aligned}$$

for all $\alpha \in \mathbb{R}^6$. If

$$\alpha_i M_{ij} \alpha_j = 0,$$

then

$$\|D(\alpha_i u_i)\|_2 = 0.$$

From Lemma 3.1, we have

$$|\alpha_i| \leqslant C(\mathcal{B}) \|D(\alpha_j u_j)\|_2, \quad i = 1, \ldots, 6,$$

which implies $\alpha_i = 0$, $i = 1, \ldots, 6$, and, consequently, M is positive definite. □

Lemma 4.2. *For each $f \in \mathcal{L}(\mathcal{D})$, there exists a solution $(u, p) \in (\mathcal{H}(\mathcal{D}) \cap \mathcal{V}(\mathcal{D}) \cap D^{2,2}(\mathcal{D})) \times D^{1,2}(\mathcal{D})$ to the problem*

$$\begin{cases} \left.\begin{aligned} & u + \bar{u} - \operatorname{div} T(u, p) = f + \overline{f} \\ & \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}, \\ u = 0 \quad \text{at } \Sigma, \\ \lim_{|x| \to \infty} (u + \bar{u})(x) = 0, \\ m u_1 = -\int_{\Sigma} T(u, p) \cdot n + m f_1, \\ I \cdot u_2 = -\int_{\Sigma} x \times T(u, p) \cdot n + I \cdot f_2. \end{cases} \tag{4.1}$$

Proof. Let $(u,p)=(\alpha_i u_i+u_f,\alpha_i p_i+p_f)$ with (u_i,p_i) solution to (2.2) and $(u_f,p_f)\in W_0^{1,2}(\mathcal{D})\cap W^{2,2}(\mathcal{D})\times D^{1,2}(\mathcal{D})$ verifying

$$\begin{cases} \left.\begin{array}{l} u_f-\operatorname{div}T(u_f,p_f)=f+\overline{f} \\ \operatorname{div}u_f=0 \end{array}\right\} \quad \text{in } \mathcal{D}, \\ u_f=0 \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}u_f(x)=0. \end{cases} \tag{4.2}$$

Then u satisfies $(4.1)_{1-4}$, for any $\alpha\in\mathbb{R}^6$. To satisfy $(4.1)_{5,6}$, we have to impose that α solves the following linear system:

$$\{S+M\}\{\alpha\}=\{\beta\}, \tag{4.3}$$

where

$$S_{ij}=\begin{cases} m\delta_{ij}, & i,j=1,2,3, \\ I_{ij}, & i,j=4,5,6, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\beta_i=\begin{cases} me_i\cdot f_1-\int_\Sigma e_i\cdot T(u_f,p_f)\cdot n, & i=1,2,3, \\ e_{i-3}\cdot I\cdot f_2-\int_\Sigma \tilde{e}_i\cdot T(u_f,p_f)\cdot n, & i=4,5,6. \end{cases}$$

Since M and S are positive definite, the matrix $S+M$ is invertible, and therefore there exists a unique $\underline{\alpha}\in\mathbb{R}^6$ that solves (4.3). Consequently, $(u,p)=(\underline{\alpha}_i u_i+u_f,\underline{\alpha}_i p_i+p_f)\in\mathcal{H}(\mathcal{D})\cap\mathcal{V}(\mathcal{D})\cap D^{2,2}(\mathcal{D})\times D^{1,2}$ solves (4.1). □

Let $P_{\mathcal{H}}$ denote the projection operator from $\mathcal{L}(\mathcal{D})$ onto $\mathcal{H}(\mathcal{D})$. We define a linear operator $A: D(A)\to\mathcal{H}(\mathcal{D})$ by

$$\begin{cases} D(A)=\mathcal{H}(\mathcal{D})\cap\mathcal{V}(\mathcal{D})\cap D^{2,2}(\mathcal{D}), \\ Au=P_{\mathcal{H}}\left[\Delta u+\frac{2}{m}\int_\Sigma D(u)\cdot n+2\left(I^{-1}\cdot\int_\Sigma x\times D(u)\cdot n\right)\times x\right]. \end{cases} \tag{4.4}$$

Notice that, since $u\in D^{2,2}(\mathcal{D})$, we have $\Delta u\in L^2(\mathcal{D})$ and $D(u)_{|\Sigma}\in W^{1/2,2}(\mathcal{D})$. The operator A is well defined because $\Delta u\in L^2(\mathcal{D})$ and

$$\frac{2}{m}\int\limits_\Sigma D(u)\cdot n+2\left(I^{-1}\cdot\int\limits_\Sigma x\times D(u)\cdot n\right)\times x\in\mathcal{R},$$

which implies that

$$\Delta u+\frac{2}{m}\int\limits_\Sigma D(u)\cdot n+2\left(I^{-1}\cdot\int\limits_\Sigma x\times D(u)\cdot n\right)\times x\in\mathcal{L}(\mathcal{D}).$$

Theorem 4.1. *The operator A defined by* (4.4) *is the generator of an analytic* (*contraction*) *semigroup in $\mathcal{H}(\mathcal{D})$.*

Proof. Let $u, v \in D(A)$. We have

$$\begin{aligned}(Au, v)_{\mathcal{H}} &= \int_{\mathcal{D}} \Delta u \cdot (v + \bar{v}) - 2v_1 \cdot \int_{\Sigma} D(u) \cdot n - 2v_2 \cdot \int_{\Sigma} x \times D(u) \cdot n \\ &= 2\int_{\Sigma} \bar{v} \cdot D(u) \cdot n - 2\int_{\mathcal{D}} D(u) : D(v) - 2v_1 \cdot \int_{\Sigma} D(u) \cdot n \\ &\quad - 2v_2 \cdot \int_{\Sigma} x \times D(u) \cdot n \\ &= -2\int_{\mathcal{D}} D(u) : D(v) = (Av, u)_{\mathcal{H}} \qquad (4.5)\end{aligned}$$

and therefore A is symmetric. In particular, we get

$$(Au, u)_{\mathcal{H}} = -2\|D(u)\|_2^2 \leqslant 0, \quad \text{for all } u \in D(A),$$

that is, A is dissipative.

Let $f \in \mathcal{H}(\mathcal{D})$ and let $(u, p) \in D(A) \times D^{1,2}(\mathcal{D})$ be the solution to system (4.1) obtained in Lemma 4.2. Notice that (4.1) can be written as

$$\begin{aligned}&u - \left[\Delta u + \frac{2}{m}\int_{\Sigma} D(u) \cdot n + 2\left(I^{-1} \cdot \int_{\Sigma} x \times D(u) \cdot n\right) \times x\right] \\ &\quad = -\left[\nabla p - \frac{1}{m}\int_{\Sigma} pn - \left(I^{-1} \cdot \int_{\Sigma} px \times n\right)\right] + f, \quad \text{in } \mathcal{L}(\mathcal{D}),\end{aligned}$$

from which we obtain the following identity in $\mathcal{H}(\mathcal{D})$:

$$u - P_{\mathcal{H}}\left[\Delta u + \frac{2}{m}\int_{\Sigma} D(u) \cdot n + 2\left(I^{-1} \cdot \int_{\Sigma} x \times D(u) \cdot n\right) \times x\right] = f.$$

We conclude that, for each $f \in \mathcal{H}(\mathcal{D})$, there exits $u \in D(A)$ such that $u - Au = f$, in $\mathcal{H}(\mathcal{D})$, that is, $\mathrm{Range}(I - A) = \mathcal{H}(\mathcal{D})$. Thus, A is maximal dissipative. By Theorem 2.1, A is the generator of analytic semigroup in $\mathcal{H}(\mathcal{D})$. □

As a consequence of Theorem 4.1, we have

Theorem 4.2. *Let* $0 < T \leqslant \infty$ *and let* $u_0 \in \mathcal{H}(\mathcal{D})$, $f \in H_{\mathrm{loc}}([0, T[\,; \mathcal{L}(\mathcal{D}))$. *Then the initial value problem* (1.3) *has a unique solution* (u, V, p) *such that*

$$\begin{aligned}&\xi, \omega \in C\big([0, T[\big) \cap C^1\big(]0, T[\big), \\ &u + V \in C\big([0, T[\,; L^2(\mathcal{D})\big) \cap C^1\big(]0, T[\,; L^2(\mathcal{D})\big) \cap C\big(]0, T[\,; W^{2,2}(\mathcal{D})\big), \\ &\nabla p \in C\big(]0, T[\,; L^2(\mathcal{D})\big).\end{aligned}$$

Proof. Let A be defined by (4.4). By Theorems 2.2 and 4.1, there exists a unique solution $u \in C([0,T[\,;\mathcal{H}(\mathcal{D})) \cap C^1(]0,T[\,;\mathcal{H}(\mathcal{D}))$ to the Cauchy problem

$$\begin{cases} \frac{du}{dt} = Au + P_{\mathcal{H}} f, & \text{in }]0,T[\,, \\ u(0) = u_0. \end{cases} \tag{4.6}$$

Since $u \in C([0,T[\,;\mathcal{H}(\mathcal{D})) \cap C^1(]0,T[\,;\mathcal{H}(\mathcal{D}))$, there exists $\xi, \omega \in C([0,T[) \cap C^1(]0,T[)$ such that $u + V \in C([0,T[\,;L^2(\mathcal{D})) \cap C^1(]0,T[\,;L^2(\mathcal{D}))$, where $V(t)(x) = \xi(t) + \omega(t) \times x$.

Let $\xi_0, \omega_0 \in \mathbb{R}^3$ be such that $u_0 + \xi_0 + \omega_0 \times x \in L^2(\mathcal{D})$. From $u(0) = u_0$, in $\mathcal{H}(\mathcal{D})$, it follows that $\xi(0) = \xi_0$ and $\omega(0) = \omega_0$.

Then, from $(4.6)_1$ and setting $u(x,t) = u(t)(x)$, we have

$$\begin{aligned} \partial_t u = {} & \Delta u + \frac{2}{m} \int_\Sigma D(u) \cdot n + 2\left(I^{-1} \cdot \int_\Sigma x \times D(u) \cdot n\right) \times x + f \\ & - P_{\mathcal{G}}\left[\Delta u + \frac{2}{m} \int_\Sigma D(u) \cdot n + 2\left(I^{-1} \cdot \int_\Sigma x \times D(u) \cdot n\right) \times x\right] \\ & - P_{\mathcal{G}} f. \end{aligned}$$

By Lemma 3.2, for each $t \in]0,T[\,$, there exists $p(\cdot,t) \in D^{1,2}(\mathcal{D})$ such that

$$\begin{aligned} & P_{\mathcal{G}}\left[\Delta u + \frac{2}{m} \int_\Sigma D(u) \cdot n + 2\left(I^{-1} \cdot \int_\Sigma x \times D(u) \cdot n\right) \times x\right] + P_{\mathcal{G}} f \\ & \quad = \nabla p + \frac{1}{m} \int_\Sigma pn + \left(I^{-1} \cdot \int_\Sigma px \times n\right) \times x. \end{aligned}$$

Then, we have

$$\begin{aligned} \partial_t u = {} & \Delta u + \frac{2}{m} \int_\Sigma D(u) \cdot n + 2\left(I^{-1} \cdot \int_\Sigma x \times D(u) \cdot n\right) \times x + f \\ & - \left[\nabla p + \frac{1}{m} \int_\Sigma pn + \left(I^{-1} \cdot \int_\Sigma px \times n\right) \times x\right], \end{aligned}$$

and therefore

$$\begin{aligned} & \partial_t(u + V) - \Delta u + \nabla p - (f + \overline{f}) \\ & \quad = \partial_t V + \frac{2}{m} \int_\Sigma D(u) \cdot n + 2\left(I^{-1} \cdot \int_\Sigma x \times D(u) \cdot n\right) \times x \\ & \qquad + \frac{1}{m} \int_\Sigma pn + \left(I^{-1} \cdot \int_\Sigma px \times n\right) \times x - \overline{f}. \end{aligned} \tag{4.7}$$

Now notice that the right-hand side of (4.7) is a rigid motion and the left-hand side of (4.7) is in $L^2(\mathcal{D})$. Since $L^2(\mathcal{D}) \cap \mathcal{R} = \emptyset$, we conclude that

$$\partial_t(u+V) = \operatorname{div} T(u,p) + f + \overline{f} \quad \text{in } \mathcal{D}$$

and

$$\partial_t V = \frac{1}{m}\int_{\Sigma} T(u,p)\cdot n + \left(I^{-1}\cdot \int_{\Sigma} x \times T(u,p)\cdot n\right) \times x + \overline{f}. \tag{4.8}$$

From Eq. (4.8) we obtain Eqs. $(1.3)_{6,7}$.

Since $\partial_t(u+V)$, $f+\overline{f} \in C(]0,T[\,;L^2(\mathcal{D}))$, we have

$$\operatorname{div} T(u,p) \in C\big(]0,T[\,;L^2(\mathcal{D})\big),$$

and by Lemma 3.3,

$$|u(t)|_{2,2} + |p(t)|_{1,2} \leqslant C(\mathcal{D})\Big(\big\|(u+V)(t)\big\|_2 + \big\|\operatorname{div} T(u,p)(t)\big\|_2 + |\xi(t)| + |\omega(t)|\Big).$$

From this estimate we easily deduce, along with the regularity properties of the solution already known and the fact that the problem is linear, that $D^2u, \nabla p \in C(]0,T[\,;L^2(\mathcal{D}))$. Finally, multiplying both sides of $(1.3)_2$ by $u+V$, integrating by parts over $\mathcal{D}$ and imposing conditions $(1.3)_{6,7}$ we get

$$2\big\|D(u)(t)\big\|_2^2 = \int_{\mathcal{D}} (u+V)(t)\cdot\big[\partial_t(u+V)(t) + (f+\overline{f})(t)\big] + m\xi(t)\cdot\left[\frac{d\xi}{dt}(t) + f_1(t)\right] + \omega(t)\cdot I\cdot\left[\frac{d\omega}{dt}(t) + f_2(t)\right]$$

which allows us to conclude that $D(u) \in C(]0,T[\,;L^2(\mathcal{D}))$. By Lemma 3.1, we have $\nabla(u+V) \in C(]0,T[\,;L^2(\mathcal{D}))$. $\square$

5. Self-propelled motion in the Stokes approximation

5.1. Existence and uniqueness of strong solution to problem (1.1)

In the next lemma we construct an extension of v_* to $\mathcal{D}$.

Lemma 5.1. *Let $v_* \in H_{\mathrm{loc}}([0,T[\,;W^{3/2,2}(\Sigma)) \cap H^1_{\mathrm{loc}}([0,T[\,;W^{1/2,2}(\Sigma))$. Then there exists an extension $\tilde{v}_*$ of v_* to $\mathcal{D}$ such that*

(i) $\operatorname{div}\tilde{v}_* = 0$ *in* $\mathcal{D}\times[0,T[$;
(ii) $\lim_{|x|\to\infty}\tilde{v}_*(x,t) = 0$ *for all* $t \in [0,T[$;
(iii) $\tilde{v}_* \in C([0,T[\,;L^2(\mathcal{D})) \cap C^1(]0,T[\,;L^2(\mathcal{D})) \cap C(]0,T[\,;W^{2,2}(\mathcal{D}))$;

(iv) $\partial_t \tilde{v}_*, \Delta \tilde{v}_* \in H_{\mathrm{loc}}([0,T[\,; L^2(\mathcal{D}))$.

Proof. Set $\Phi(t) = \int_\Sigma v_*(x,t)\cdot n_x\, d\sigma_x$ and $\Theta(x,t) = \Phi(t)\nabla\mathcal{E}(x)$, where $\mathcal{E}$ is the fundamental solution to the Laplace equation. Then

$$\operatorname{div}\Theta = 0 \quad \text{in } \mathcal{D}\times[0,T[\,,$$
$$\int_\Sigma \Theta(x,t)\cdot n_x\, d\sigma_x = \Phi(t).$$

Putting $w_* = v_* - \Theta_{|\Sigma}$, it follows that $\int_\Sigma w_*(x,t)\cdot n_x d\sigma_x = 0$, for all $t \in [0,T[$.

Let $R > \delta(\mathcal{B})$ be fixed. For each $t \in \,]0,T[$, let $\overline{v_*}(t) \in W^{2,2}(\mathcal{D})$ be a solenoidal field vanishing outside $\mathcal{D}_R$, that equals $w_*(t)$ on Σ. The existence of $\overline{v_*}(t)$ enjoying these properties is ensured by Lemma 2.1. Moreover, we have

$$\|\overline{v_*}(t)\|_{W^{2,2}(\mathcal{D})} \leqslant C(\mathcal{B})\|v_*(t)\|_{W^{3/2,2}(\Sigma)} \tag{5.1}$$

for all $t \in [0,T[$. Since $\overline{v_*}$ can be constructed using explicit representation formulas, it is easily seen that $\partial_t\overline{v_*}(t) \in W^{1,2}(\mathcal{D})$, for all $t \in \,]0,T[$, and

$$\|\partial_t\overline{v_*}(t)\|_{W^{1,2}(\mathcal{D})} \leqslant C(\mathcal{B})\|\partial_t v_*(t)\|_{W^{1/2,2}(\Sigma)}. \tag{5.2}$$

We define the extension $\tilde{v}_*$ by $\tilde{v}_* = \overline{v_*} + \Theta$. Due to the linearity of the problem and estimates (5.1) and (5.2), we conclude that

$$\tilde{v}_* \in C\big([0,T[\,; L^2(\mathcal{D})\big) \cap C^1\big(]0,T[\,; L^2(\mathcal{D})\big) \cap C\big(]0,T[\,; W^{2,2}(\mathcal{D})\big)$$

and

$$\partial_t\tilde{v}_*, \Delta\tilde{v}_* \in H_{\mathrm{loc}}\big([0,T[\,; L^2(\mathcal{D})\big). \qquad \square$$

We are now in position to prove

Theorem 5.1. *Let $\mathcal{B}$ be a rigid body with boundary Σ of class C^2 and let $0 < T \leqslant \infty$. Then for any $v_* \in H_{\mathrm{loc}}([0,T[\,; W^{3/2,2}(\Sigma)) \cap H^1_{\mathrm{loc}}([0,T[\,; W^{1/2,2}(\Sigma))$, $V_0 \in \mathcal{R}$ and $v_0 \in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$ satisfying*

$$\operatorname{div} v_0 = 0,$$
$$v_0\cdot n = v_*(\cdot,0)\cdot n \quad \text{at } \Sigma,$$
$$v_0 + V_0 \in L^2(\mathcal{D})$$

there exists a unique solution (v, V, p) to (1.1) such that

$$\xi,\omega \in C\big([0,T[\big) \cap C^1\big(]0,T[\big),$$
$$v + V \in C\big([0,T[\,; L^2(\mathcal{D})\big) \cap C^1\big(]0,T[\,; L^2(\mathcal{D})\big) \cap C\big(]0,T[\,; W^{2,2}(\mathcal{D})\big),$$
$$\nabla p \in C\big(]0,T[\,; L^2(\mathcal{D})\big).$$

Proof. Decomposing v as

$$v = u + \tilde{v}_*$$

with $\tilde{v}_*$ given by Lemma 5.1, we can rewrite (1.1) in the form (1.3) with

$$\begin{cases} \left.\begin{array}{l} \partial_t(u+V) = \operatorname{div} T(u,p) + \Delta\tilde{v}_* - \partial_t\tilde{v}_* \\ \operatorname{div} u = 0 \end{array}\right\} \quad \text{in } \mathcal{D}\times\,]0,T[\,, \\ u = 0 \quad \text{at } \Sigma, \\ \lim_{|x|\to\infty}(u(x,t)+V(x,t)) = 0, \quad t\in[0,T[\,, \\ m\frac{d\xi}{dt} = -\int_\Sigma T(u,p)\cdot n - 2\int_\Sigma D(\tilde{v}_*)\cdot n, \\ I\cdot\frac{d\omega}{dt} = -\int_\Sigma x\times T(u,p)\cdot n - 2\int_\Sigma x\times D(\tilde{v}_*)\cdot n, \\ u(x,0) = v_0(x) - \tilde{v}_*(x,0), \quad c\in\mathcal{D}, \\ \xi(0) = \xi_0, \qquad \omega(0) = \omega_0. \end{cases} \tag{5.3}$$

Since

$$\Delta\tilde{v}_* - \partial_t\tilde{v}_* + \frac{2}{m}\int_\Sigma D(\tilde{v}_*)\cdot n + 2I^{-1}\left(\int_\Sigma x\times D(\tilde{v}_*)\cdot n\right)\times x$$
$$\in H_{\mathrm{loc}}\big([0,T[\,;\mathcal{L}(\mathcal{D})\big),$$

and $v_0 - \tilde{v}_*(\cdot,0)\in\mathcal{H}(\mathcal{D})$, Theorem 4.2 guarantees the existence and uniqueness of (u,V,p) such that

$$\xi,\omega\in C\big([0,T[\big)\cap C^1\big(]0,T[\big),$$
$$u+V\in C\big([0,T[\,;L^2(\mathcal{D})\big)\cap C^1\big(]0,T[\,;L^2(\mathcal{D})\big)\cap C\big(]0,T[\,;W^{2,2}(\mathcal{D})\big),$$
$$\nabla p\in C\big(]0,T[\,;L^2(\mathcal{D})\big),$$

and since $\tilde{v}_*\in C([0,T[\,;L^2(\mathcal{D}))\cap C^1(]0,T[\,;L^2(\mathcal{D}))\cap C(]0,T[\,;W^{2,2}(\mathcal{D}))$, the proof is complete. □

5.2. Existence, uniqueness and L^q-estimates of solution to problem (1.2)

Let T be the 6×6 matrix defined by

$$T_{ij} = \int_\Sigma \tilde{e}_j\cdot T(H_i,P_i)\cdot n, \quad i,j = 1,\ldots,6,$$

where (H_i,P_i) are the auxiliary fields solving (2.1). Then we have (see [15])

Lemma 5.2. *T is symmetric and positive definite.*

Define

$$g_i = T(H_i,P_i)\cdot n_{|\Sigma}, \quad i = 1,\ldots,6.$$

Set $(v, p) = (\alpha_i H_i + u, \alpha_i P_i + \pi)$, with (H_i, P_i), (u, π) as in Lemmas 2.2 and 2.4, and $\alpha \in \mathbb{R}^6$. Then v satisfies $(1.2)_{1-4}$, for any $\alpha \in \mathbb{R}^6$. To satisfy $(1.2)_{5,6}$, we have to impose that α solves the following linear system:

$$\{T\}\{\alpha\} = \{\beta\}, \tag{5.4}$$

where

$$\beta_i = -\int_{\Sigma} \tilde{e}_i \cdot T(u, p) \cdot n = -\int_{\Sigma} v_* \cdot g_i, \quad i = 1, \dots, 6. \tag{5.5}$$

The second identity in (5.5) is obtained by multiplying $(2.1)_1$ by u and $(2.3)_1$ by $H_i + \tilde{e}_i$, and integrating by parts over $\mathcal{D}$. Since T is positive definite, it is invertible, and therefore there exists a unique $\alpha \in \mathbb{R}^6$ such that (5.4) holds. Consequently, $(v, p) = (\alpha_i H_i + u, \alpha_i P_i + \pi)$ with α given by (5.4) solves (1.2).

In [10] it is proved that the set $\{g_1, \dots, g_6\} \subset L^r(\Sigma)$, $r > 3/2$, is linearly independent and the *thrust space*

$$\mathcal{T}(\mathcal{B}) := \operatorname{span}\{g_1, \dots, g_6\}$$

is introduced. In $\mathcal{T}(\mathcal{B})$ we consider the following norm:

$$\|g\| = \left\| \sum_{i=1}^{6} \alpha_i g_i \right\| := \sum_{i=1}^{6} |\alpha_i|.$$

We denote by $\mathbb{P}$ the projection operator in $\mathcal{T}(\mathcal{B}) \subset L^2(\Sigma)$.

Theorem 5.2. *Let $v_* \in W^{2-1/q,q}(\Sigma)$, $3 < q < \infty$. Then problem* (1.2) *admits one and only one solution (v, p, V) such that $v, p \in C^\infty(\mathcal{D})$, and*

$$(v + V, p) \in \left(L^r(\mathcal{D}) \cap D^{1,s_1}(\mathcal{D}) \cap D^{2,\tau}(\mathcal{D})\right) \times \left(L^{s_2}(\mathcal{D}) \cap D^{1,\tau}(\mathcal{D})\right),$$

for $r \in]3/2, \infty]$, $s_1 \in]6/5, \infty]$, $s_2 \in]1, \infty]$ and $\tau \in]1, q]$, and the following estimates hold:

$$C_1\|\mathbb{P}(v_*)\| \leqslant |\xi| + |\omega| \leqslant C_2\|\mathbb{P}(v_*)\|,$$
$$\begin{aligned}&\left\|\left(1 + |x|^2\right)(v + V)\right\|_\infty + \left\|\nabla(v + V)\right\|_\infty + \left\|\left(1 + |x|^3\right)p\right\|_\infty \\ &\qquad + \|v + V\|_r + |v + V|_{1,s_1} + |v|_{2,\tau} + \|p\|_{s_2} + |p|_{1,\tau} \\ &\quad \leqslant C_3\|v_*\|_{2-1/q,q(\Sigma)},\end{aligned}$$

with $C_i = C_i(\mathcal{B})$, $i = 1, 2$, $C_3 = C_3(\mathcal{B}, q, r, s, \tau)$.

Proof. From (5.4), we have

$$\|T\|^{-1}|\beta| \leqslant |\alpha| \leqslant \|T^{-1}\||\beta|,$$

and since β verifies $K_1(\mathcal{B})\|\mathbb{P}(v_*)\| \leqslant |\beta| \leqslant K_2(\mathcal{B})\|\mathbb{P}(v_*)\|$, it follows that

$$C_1(\mathcal{B})\|\mathbb{P}(v_*)\| \leqslant |\xi| + |\omega| \leqslant C_2(\mathcal{B})\|\mathbb{P}(v_*)\|. \tag{5.6}$$

From Lemmas 2.2 and 2.4 and from (5.6), we get

$$\begin{aligned}&\big\|\big(1+|x|\big)(v+V)\big\|_\infty+\big\|\nabla(v+V)\big\|_\infty+\|p\|_\infty\\&\qquad+\|v+V\|_r+|v+V|_{1,s}+|v|_{2,\tau}+\|p\|_s+|p|_{1,\tau}\\&\quad\leqslant C_3\|v_*\|_{2-1/q,q(\Sigma)},\end{aligned}\tag{5.7}$$

for $r\in]3,\infty]$, $s\in]3/2,\infty]$ and $\tau\in]1,q]$, with $C=C(\mathcal{B},q,r,s,\tau)$. Next, we prove the following inequalities:

$$\big\|\big(1+|x|^2\big)(v+V)\big\|_\infty+\big\|\big(1+|x|^3\big)p\big\|_\infty\leqslant C(\mathcal{B})\|v_*\|_{2-1/q,q,\Sigma}.\tag{5.8}$$

From the integral representation of (v,V,p) (see Theorem V.3.3 in [7]) and in view of the self-propelling condition $\int_\Sigma T(v,p)\cdot n=0$, we can write

$$\begin{aligned}(v+V)_j(x)&=-\int_\Sigma\big[\mathcal{U}_{ij}(x-y)T_{il}(v,p)(y)\\&\qquad-(v_{*i}+V_i)(y)T_{il}(\mathcal{U}_j,\mathcal{P}_j)(x-y)\big]n_l(y)\,d\sigma_y\\&=-\int_\Sigma\big[\big(\mathcal{U}_{ij}(x-y)-\mathcal{U}_{ij}(x)\big)T_{il}(v,p)(y)\\&\qquad-(v_{*i}+V_i)(y)T_{il}(\mathcal{U}_j,\mathcal{P}_j)(x-y)\big]n_l(y)\,d\sigma_y,\\p(x)&=\int_\Sigma\bigg[\mathcal{P}_i(x-y)T_{il}(v,p)(y)-2(v_{*i}+V_i)(y)\frac{\partial\mathcal{P}_l(x-y)}{\partial y_i}\bigg]n_l(y)\,d\sigma_y\\&=\int_\Sigma\bigg[\big(\mathcal{P}_i(x-y)-\mathcal{P}_i(x)\big)T_{il}(v,p)(y)\\&\qquad-2(v_{*i}+V_i)(y)\frac{\partial\mathcal{P}_l(x-y)}{\partial y_i}\bigg]n_l(y)\,d\sigma_y,\end{aligned}$$

for $j=1,2,3$. The pair $(\mathcal{U},\mathcal{P})=(\mathcal{U}_{ij},\mathcal{P}_i)$, $i,j=1,2,3$, is the fundamental solution of the Stokes equation

$$\mathcal{U}_{ij}(x)=-\frac{1}{8\pi}\bigg(\frac{\delta_{ij}}{|x|}+\frac{x_ix_j}{|x|^3}\bigg),\qquad\mathcal{P}_i(x)=\frac{1}{4\pi}\frac{x_i}{|x|^3},$$

which satisfies

$$\begin{aligned}&\big|\mathcal{U}_{ij}(x-y)-\mathcal{U}_{ij}(x)\big|,\ \big|T_{il}(\mathcal{U}_j,\mathcal{P}_j)(x-y)\big|=O\big(|x|^{-2}\big),\\&\big|\mathcal{P}_i(x-y)-\mathcal{P}_i(x)\big|,\ \bigg|\frac{\partial\mathcal{P}_l(x-y)}{\partial y_i}\bigg|=O\big(|x|^{-3}\big),\end{aligned}$$

uniformly with respect to y in a bounded set. Thus,

$$\begin{aligned}&\big\|\big(1+|x|^2\big)(v+V)\big\|_{\infty,\mathcal{D}^{2\delta(\mathcal{B})}}+\big\|\big(1+|x|^3\big)p\big\|_{\infty,\mathcal{D}^{2\delta(\mathcal{B})}}\\&\quad\leqslant C(\mathcal{B})\int\limits_{\Sigma}\big(|T(v,p)|+|v_*+V|\big),\end{aligned}$$

and by estimates (5.6), (5.7) and trace theorem, we get

$$\begin{aligned}&\big\|\big(1+|x|^2\big)(v+V)\big\|_{\infty,\mathcal{D}^{2\delta(\mathcal{B})}}+\big\|\big(1+|x|^3\big)p\big\|_{\infty,\mathcal{D}^{2\delta(\mathcal{B})}}\\&\quad\leqslant C(\mathcal{B})\|v_*\|_{2-1/q,q(\Sigma)}.\end{aligned}$$

From (5.7) we easily conclude that

$$\begin{aligned}&\big\|\big(1+|x|^2\big)(v+V)\big\|_{\infty,\mathcal{D}_{2\delta(\mathcal{B})}}+\big\|\big(1+|x|^3\big)p\big\|_{\infty,\mathcal{D}_{2\delta(\mathcal{B})}}\\&\quad\leqslant C(\mathcal{B})\|v_*\|_{2-1/q,q(\Sigma)},\end{aligned}$$

and therefore (5.8) hold true. This implies that $(v+V)\in L^r(\mathcal{D})$, $r>3/2$, $p\in L^s(\mathcal{D})$, $s>1$, with

$$\|v+V\|_r+\|p\|_s\leqslant C(\mathcal{B},r,s)\|v_*\|_{2-1/q,q(\Sigma)}.$$

Let $R_0>2\delta(\mathcal{B})$. From (5.7), we get

$$|v+V|_{1,s,\mathcal{D}_{R_0}}\leqslant C(\mathcal{B},R_0,s,q)\|v_*\|_{2-1/q,q(\Sigma)},\quad s\geqslant 1.$$

Let ψ be a smooth cut-off function such that $0\leqslant\psi\leqslant 1$, $\psi(x)=1$ for $|x|\geqslant R_0$ and $\psi=0$ in a neighborhood of Σ. Now, we consider the function $\psi(v+V)$ defined in $\mathbb{R}^3$, which verifies $\psi(v+V)\in D^{2,\tau}(\mathbb{R}^3)$, $\tau\in]1,q]$. Using the estimates for $v+V$ already obtained, we get

$$\big|\psi(v+V)\big|_{2,\tau,\mathbb{R}^3}\leqslant C(\mathcal{B},\tau)\big(\|\psi\|_{2,\infty,\mathbb{R}^3}\|v+V\|_{1,\tau,\mathcal{D}_{R_0}}+|v+V|_{2,\tau,\mathcal{D}}\big).$$

By a multiplicative inequality due to Nirenberg, see [7], we have

$$|v+V|_{1,s,\mathcal{D}^{R_0}}\leqslant\big|\psi(v+V)\big|_{1,s,\mathbb{R}^3}\leqslant C\|v+V\|_{r,\mathcal{D}}^{1-a}\big|\psi(v+V)\big|_{2,t,\mathbb{R}^3}^{a},$$

where

$$\frac{1}{s}=\frac{1}{3}+a\left(\frac{1}{r}-\frac{2}{3}\right)+(1-a)\frac{1}{q},\quad\text{with }\frac{1}{2}\leqslant a\leqslant 1,$$

from which, along with the summability properties already derived, we conclude, with $a=1/2$, that $v+V\in D^{1,s}$, $s>6/5$, and

$$|v+V|_{1,s}\leqslant C(\mathcal{B},s,q)\|v_*\|_{2-1/q,q(\Sigma)}.$$

Concerning the uniqueness, suppose that (v_i,V_i,p_i), $i=1,2$, are two different solutions satisfying the summability properties stated in the theorem. Then $(v, V,p)=(v_1-v_2,V_1-V_2,p_1-p_2)$ satisfies

$$\begin{cases}\left.\begin{array}{l}\operatorname{div}T(v,p)=0\\ \operatorname{div}v=0\end{array}\right\}\quad\text{in }\mathcal{D},\\ v_{|\Sigma}=0,\\ \lim_{|x|\to\infty}(v+V)(x)=0,\\ \int_\Sigma T(v,p)\cdot n=\int_\Sigma x\times T(v,p)\cdot n=0.\end{cases}\tag{5.9}$$

Multiplying both sides of $(5.9)_1$ by $v+V$ and integrating by parts over $\mathcal{D}$, we get

$$\xi\cdot\int_{\Sigma} T(v,p)\cdot n+\omega\cdot\int_{\Sigma} x\times T(v,p)\cdot n=2\|D(v)\|_2^2,$$

and due to the self-propelling conditions $(5.9)_5$, we conclude that

$$\|D(v)\|_2=0.$$

But, by Lemma 3.1, this implies that $\xi=\omega=0$ and $\nabla v=0$. Since $(5.9)_{3,4}$ are satisfied if and only if $v=0$, we conclude that $(v,V,p)=(0,0,0)$. □

From (5.6), we conclude that $V\neq 0$ if and only if $\mathbb{P}(v_*)\neq 0$. Thus, to propel $\mathcal{B}$ with a *nonzero* velocity, we should prescribe boundary velocities such that $\mathbb{P}(v_*)\neq 0$. For a more detailed discussion on the relation between V and v_*, see [10].

It is well known that if $\mathcal{B}$ moves in $\mathcal{L}$ with velocity $V(x)=\xi+\omega\times x$ and $\int_\Sigma T(v,p)\cdot n\neq 0$, then (v,p) only verifies

$$(v+V,p)\in\big(L^s(\mathcal{D})\cap D^{1,r}(\mathcal{D})\cap D^{2,\tau}(\mathcal{D})\big)\times\big(L^r(\mathcal{D})\cap D^{1,\tau}(\mathcal{D})\big)$$

with $s\in\,]3,\infty]$, $r\in\,]3/2,\infty]$, $\tau\in\,]1,q]$.

5.3. *Attainability of steady self-propelled slow motion*

Our objective is to show that a steady self-propelled solution (1.2), which we denote now by $(v_\infty,p_\infty,V_\infty)$ with $V_\infty(x)=\xi_\infty+\omega_\infty\times x$, can be obtained as the limit, as $t\to\infty$, of a nonstationary solution

$$\begin{cases}\left.\begin{array}{l}\partial_t(v+V)=\operatorname{div}T(v,p)\\ \operatorname{div}v=0\end{array}\right\}\quad \text{in } \mathcal{D}\times(0,\infty),\\ v=w_*\quad \text{at } \Sigma\times(0,\infty),\\ \lim_{|x|\to\infty}(v(x,t)+V(x,t))=0,\quad t\in(0,\infty),\\ m\frac{d\xi}{dt}=-\int_\Sigma T(v,p)\cdot n,\\ I\cdot\frac{d\omega}{dt}=-\int_\Sigma x\times T(v,p)\cdot n,\\ v(x,0)=0,\quad x\in\mathcal{D},\\ \xi(0)=0,\qquad \omega(0)=0.\end{cases}\tag{5.10}$$

Theorem 5.3. *Let $w_*(x,t)=\psi(t)v_*(x)$, where ψ is a smooth real function that is zero for $t\leqslant 0$ and is one for $t\geqslant t_0$, and $v_*\in W^{2-1/q,q}(\Sigma)$, $3<q<\infty$. Then problem* (5.10) *admits a unique strong solution (v,p,ξ,ω) such that*

$$\big\|\partial_t(v+V)(t)\big\|_2,\left|\frac{d\xi}{dt}\right|,\left|\frac{d\omega}{dt}\right|=O(t^{-1}),$$

$$\big\|D\big(v(t)-v_\infty\big)\big\|_2,\big|\xi(t)-\xi_\infty\big|,\big|\omega(t)-\omega_\infty\big|=O(t^{-1/2}),$$

$$\left|v(t)-v_\infty\right|_{2,2},\ \sup_{x\in\mathcal{D}}\left|v(x,t)-V_\infty(x,t)\right|,\ \left|p(t)-p_\infty\right|_{1,2}=O(t^{-1/2}),$$

as $t\to\infty$.

Proof. Setting

$$\begin{aligned}
&u(x,t)=v(x,t)-\psi(t)v_\infty(x),\\
&\pi(x,t)=p(x,t)-\psi(t)p_\infty(x),\\
&\zeta(t)=\xi(t)-\psi(t)\xi_\infty,\\
&\Omega(t)=\omega(t)-\psi(t)\omega_\infty,\\
&U(t)=\zeta(t)+\Omega(t)\times x,
\end{aligned}$$

it is easily seen that (v,p,ξ,ω) is a solution to problem (5.10) if and only if (u,π,ζ,Ω) solves

$$\begin{cases}
\left.\begin{array}{l}\partial_t(u+U)=\operatorname{div}T(u,\pi)+\psi'(v_\infty+V_\infty)\\ \operatorname{div}u=0\end{array}\right\} \quad \text{in } \mathcal{D}\times(0,\infty),\\
u=0 \quad \text{at } \Sigma\times(0,\infty),\\
\lim_{|x|\to\infty}(u(x,t)+U(t))=0, \quad t\in(0,\infty),\\
m\frac{d\zeta}{dt}=-\int_\Sigma T(u,\pi)\cdot n+m\psi'\xi_\infty,\\
I\cdot\frac{d\Omega}{dt}=-\int_\Sigma x\times T(u,\pi)\cdot n+\psi' I\cdot\omega_\infty,\\
u(x,0)=0, \quad x\in\mathcal{D},\\
\zeta(0)=0, \qquad \Omega(0)=0.
\end{cases} \tag{5.11}$$

The existence and uniqueness of strong solution to this problem are given by Theorem 4.2.

Recalling estimates in Theorems 2.2 and 5.2, we have

$$\begin{aligned}
\|u(t)\|_{\mathcal{H}(\mathcal{D})}&\leqslant\|\psi' P_{\mathcal{H}}v_\infty\|_{L^1(0,\infty,\mathcal{H}(\mathcal{D}))}\\
&\leqslant\left(\|v_\infty+V_\infty\|_2^2+m|\xi_\infty|^2+\omega_\infty\cdot I\cdot\omega_\infty\right)^{1/2}I(\psi')\\
&\leqslant C(\mathcal{B},q)I(\psi')\|v_*\|_{2-1/q,q,\Sigma},
\end{aligned}$$

for all $t>0$, where $I(\psi')=\int_0^{t_0}|\psi'(s)|\,ds$. For $t>t_0$, it is $\psi'=0$, and therefore

$$\left\|\frac{du}{dt}(t)\right\|_{\mathcal{H}(\mathcal{D})}\leqslant\frac{\|u(t_0)\|_{\mathcal{H}(\mathcal{D})}}{t-t_0}\leqslant\frac{C(\mathcal{B},q)I(\psi')\|v_*\|_{2-1/q,q,\Sigma}}{t-t_0}, \tag{5.12}$$

which allows us to conclude that

$$\left\|\partial_t(u+U)\right\|_2,\ \left|\frac{d\zeta}{dt}\right|,\ \left|\frac{d\Omega}{dt}\right|=O(t^{-1}), \quad t\to\infty. \tag{5.13}$$

Multiplying both sides of $(5.11)_1$ by $u+U$, integrating by parts over $\mathcal{D}$ and imposing conditions $(5.11)_{5,6}$ we get

$$\frac{1}{2}\frac{d}{dt}\left[\|u+U\|_2^2+m|\zeta|^2+\Omega\cdot I\cdot\Omega\right]+2\|D(u)\|_2^2=0, \tag{5.14}$$

and using estimate (5.12), we deduce that

$$\begin{aligned}\|D(u)\|_2^2 &\leqslant \frac{1}{2}\big[\|u+U\|_2^2 + m|\zeta|^2 + \Omega\cdot I\cdot\Omega\big]^{1/2}\\ &\quad\times\left[\big\|\partial_t(u+U)\big\|_2^2 + m\left|\frac{d\zeta}{dt}\right|^2 + \frac{d\Omega}{dt}\cdot I\cdot\frac{d\Omega}{dt}\right]^{1/2}\\ &\leqslant C\left[\big\|\partial_t(u+U)\big\|_2^2 + m\left|\frac{d\zeta}{dt}\right|^2 + \frac{d\Omega}{dt}\cdot I\cdot\frac{d\Omega}{dt}\right]^{1/2},\end{aligned}$$

where $C = C(\mathcal{B}, q)\|v_*\|_{2-1/q,q(\Sigma)} I(\psi')$, and therefore, taking into account (5.13), we get

$$|\zeta|, |\Omega|, \|D(u)\|_2 = O(t^{-1/2}), \quad \text{as } t\to\infty.$$

Since, for $t > t_0$, it is $\|\operatorname{div} T(u,\pi)\|_2 = \|\partial_t(u+U)\|_2$, and, from Lemma 3.3, $|u|_{2,2}, |\pi|_{1,2} \leqslant C(\mathcal{B})(\|\partial_t(u+U)\|_2 + \|D(u)\|_2)$, then

$$|u|_{2,2}, |\pi|_{1,2} = O(t^{-1/2}), \quad \text{as } t\to\infty.$$

Finally, since $\|u+U\|_\infty \leqslant C(\mathcal{B})\|u+U\|_{1,6} \leqslant C(\|D(u)\|_2 + \|D^2u\|_2)$, we get

$$\|u+U\|_\infty = O(t^{-1/2}), \quad \text{as } t\to\infty. \qquad \square$$

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